Scalar and Vector Field

A physical quantity, expressed as a continuous function of the position of a point in a region of space is referred to as a point function (or function of position). The region of space concerned is known as a scalar field or a vector field according as the physical quantity in question is a scalar or a vector one expressed as a continuous single-valued function $\phi(x,y,z)$ or $\vec{F}(x,y,z)$ respectively, of position in that region.

Example of scalar fields may be mentioned the distribution of temperature or magnetic and electrostatic field potentials in given region of space.

And example of vector fields, we have distribution of magnetic and electric intensity or the distribution of velocity in moving (continuous) fluid.

Vector Calculus

Partial Derivative - Gradient

Partial derivative - The differentiation of a function such as $\phi(x_1, x_2, x_3 \ldots)$ of two or more independent variables, x_1, x_2, x_3 etc. with respect to one of them, keeping the other constant is called partial differentiation. The derivative thus obtained are reffered to as partial derivatives and are denoted by the symbol δ instead of d.

Thus if we have a continuously differential scalar point function $\phi(x, y, z)$ i.e, a function of coordinates x, y, z, its partial derivatives along three coordinates axes are

$$\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \text{ and } \frac{\partial \phi}{\partial z}$$

Gradient :- The vector function $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

So differential $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$$= \left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

or, $d\phi = dr \text{ grad } \phi$

It can be easily be shown that grade ϕ at any point is quite independent of the choice of coordinates axes. It follows therefore that the gradient of a scalar point function is a vector point function.

~~If~~ if ϕ is constant then partial derivative i.e $\partial\phi/\partial x$, $\partial\phi/\partial y$ and $\partial\phi/\partial z$ will obviously be zero.

Thus grad ϕ will also be zero, and if grad $\phi = 0$ then the partial derivatives are all zero and hence function ϕ is a constant. Thus grade $\phi = 0$ only if ϕ be constant.

In vector algebra, a differential operator is denoted by $\vec{\nabla}$ (i.e inverted Δ). It is called 'del'. It operates distributively and is formally assumed to ~~be~~ have the character of a vector.

Since it is a vector quantity, its product $\vec{\nabla}\phi$ with a scalar ϕ is a vector.

If we choose Cartesian orthogonal coordinates, we may write

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

So that,

$$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \vec{\nabla}\phi \text{, a vector}$$

Divergence and Curl

From above we have seen how from a scalar point function ϕ we can obtain a vector point function $\overrightarrow{\text{grade}}\ \phi$. Similarly, from a vector point function f we can obtain two more point functions: scalar one, called divergence and a vector one called curl (or rotation).

Thus if $\vec{f}$ ~~vectors~~ represents a vector field, i.e, if $\vec{f}$ be a continuously differentiable vector point function, the function

$$\hat{i}\frac{\partial \vec{f}}{\partial x} + \hat{j}\frac{\partial \vec{f}}{\partial y} + \hat{k}\frac{\partial \vec{f}}{\partial z}$$

which is a scalar, is called divergence of $\vec{f}$ and is written as div $\vec{f}$. And the function

$$\hat{i}\times\frac{\partial \vec{f}}{\partial x} + \hat{j}\times\frac{\partial \vec{f}}{\partial y} + \hat{k}\times\frac{\partial \vec{f}}{\partial z}$$

which is a vector, is called curl of $\vec{f}$ and is written as curl $\vec{f}$.